**General Features of the Distribution P(g)**

Want to take a broad view of the scaling function and consider what the implications of such a function are, and how it fits within the framework already established.

**P(g) in z >> L limit, and Oseledec’s theorem**

So we have:



And the xm here are related to the Lyapunov exponents via xm = νmz. The transfer matrix obeys a multiplicative property recursion relation which satisifies the Markov and Martingale properties. As such we may consider the transfer matrix to be a ‘random’ function of Lz. In the limit of weak disorder it would be a continuous function, while for strong disorder, discrete. According to Oseledec’s theorem, the eigenvalues, Qn, of Q = MM† ought to approach a random variable:



where the νn = 1/ξn are called the Lyapunov exponents. These can be considered a localization length I guess, for each channel. And these would differ because the channels each have different longitudinal velocities, and ξ depends on vz. Of course there is only ‘one’ localization length at a given energy, so perhaps the majority of the statistical properties are determined by the first (smallest) lyapunov exponent, which is the largest localization length. The exponents are distributed as the following random variable:



where W = N(0,1). And so then xn is distributed as:



[this conjecture of mine is repeated in a comment in Markos paper]. So what are Qn distributed as? Or more instructive, how are λn distributed (and it’s of the same order as Qn)? So λn itself will converge to a ln-normal-ish distribution. Let’s be explicit, and propose that νn is normally distributed with std σn/√z, and average n: Then what is P(λ = ezν)?



Really should’ve written z → z/ℓ, for unit’s sake i think:



So we see that on general grounds we expect lnλ’s to be normally distributed with an average and mean that grow linearly with length. We cannot conclude anything yet about how separated the λ’s are, because we do no know how the <νn>’s are distributed. Well actually, even and σn can be derived using just the ν evolution equation (involving the K-matrix). And even besides that we know that it is z-independent. So at least we can say that the mean spacing is:



and the average width is:



And the ratio is:



and so regardless of what those two constants are, the Δlnλ’s are bound to become well separated enough. So then eventually we must be able to say, g = 1/(1+λ) ~ 1/λ1 → lnλ1 = -lng. And so we expect lng to also be normally distributed. Does anything change by going to large disorder? Doesn’t look like it. Does this mean that the DMPK describes large disorder too, in the z >> W limit? It seems so (or at least up to this order). One could sortof argue that the e-‘s become isotropically distributed for z >> W due to repeated reflections, but one could also argue that by the time z is large enough, there is hardly anything out there, and so unlikely to become equally distributed? Not sure. Looking ahead a little bit, we know that Kmm reverts to an isotropic distribution pretty quickly, and ν1 goes first. And so numerical evidence indicates it does go to Q1D. Also, evidence from the ν-K equation concurs. Calculating <νn> and <νn>2 direct from the equation imply a proportionality ratio of <ν1>2 = 1/2z (and <lng>2/<lng>1 = 1), regardless of disorder.

**P(g) distribution in z ~ L limit?**

But let’s back up. So how far can we back z up without violating the assumptions of our previous discussion? I guess z > ξ. And ξn­ = ℓ/νn (these two statements are no more than basically zνn >> 1, which is when the exponential is ‘large’). And so need z > ξ1, where ξ1 = ℓ/ν1 is the maximum localization length. In Q1D, weak disorder at least, this would be ξ1 = ℓ/ν1 ~ Nℓ = W2kF2ℓ = W2ℓ/λF2 = (W/λF)2ℓ. How does this compare to W? Well,



So ξ1 >> W basically always. So when z ~ W or less, we’re definitely not in the localized regime. How does the lnλ formula look in this case? Even if we were to assume that the ν’s have settled into δ’s, and the λ’s into ln-normalish distributions, the separation between the neighboring lnλ’s would be, as aforementioned,



In Q1D, this would go as:



And so this wouldn’t be so valid for z ~ W.. But what about higher disorder. What changes, presumably, is that νmin increases to a finite value, even in the large N limit. This is because we expect the largest localization length to be ~ ℓ itself, as the path integral estimates of σ rather indicated. So then, ξmax ~ ℓ or so. So we see that we don’t even allow ourselves to get into an Ohm’s law-like region: it’s exponentially decreasing after a few ℓ’s). So then,



**P(g) in z < L limit?**

But this will be quite small even for z = ℓ. So we cannot assume they are separated at all. OK, so what about weak disorder and small lengths? Well, the weak disorder scenario, typically assumes λ is:



where W is N(0,1). So now



(presuming we can say that the λ’s are independently distributed). Then we need <λi>2 to be ~ 1/N. Well, this would be presuming too much at this point.

**Lyapunov exponent distribution ρ(ν) in various regimes**

The ν’s will be distributed with some density: ρ(ν) = <Σδ(ν-νn­­)>.  The minimum ν would define the localization length, via ξ = 1/νmin?   Note this ought to depend on energy and disorder of course.  In the weak disorder Q1D regime, these are uniformly distributed between 0 and N.   And this would imply ξQ1D ~ Nℓ, as it is.  In 3D, for weak disorder, we would expect that like in Q1D, which should have νmin ~ 1/N.  But for large disorder, we’d expect ξ3D ~ ℓ, and so νmin ~ 1, instead.  Thus, in the thermodynamic limit, we expect a gap to open up in the exponent spectrum.  And indeed νmin may be taken as an order parameter of the transition.

**Close Lyapunov exponent correlations**

According to Markos’ paper, we expect P(Δλ) ~ (Δλ)β as Δλ → 0.

**Tr(λ)**

But then when calculate <Trλ> we get unequivocal exponential growth, regardless of how L relates to z. Whereas we expect that λ = (1-T)/T ~ 1-T (in metallic state), and so Trλ ought to go as N(1-T). But we get:



which is roughly,

, 

in the white noise, Q1D model respectively. This seems to contradict fact that sample should scale as metal in the limit of large length, for weak disorder. I think I get it finally. I suppose that most λ’s are actually really small. But there is a λN or something that is always blowing up, and will dominate the trace regardless. Perhaps then, as Lz = L is increasing, more channels are being added and being pushed out. Most of them are going to be centered around the origin, but a few – the ones that have been pushed out the longest – will be exponentially separated from the origin. This won’t affect <g>, but it does affect Trλ. This coheres with the fact that the λ’s are like exp(νnz) ~ exp(nz/N). And so as the ones that dominate the conductance, say n = 4, will be given by λ4 = exp(4Lz/L2), and so will be compressed towards the origin as the cube expands. But, say, the one at the middle of the spectrum, n = N/2, will go as λN/2 = exp(z/2), and so will always be pushed out as Lz = L expands.

So note that since <Tr(λ)> grows in metallic regime, and certainly in insulating regime too, it would seem likely that it does at critical point. And so it would seem that not all statistical quantities will remain invariant at the critical point. And so this would imply P(λ) is itself not invariant at the critical point. And so P(g) will not be either? Perhaps multiple moments will be.

Or perhaps the entire distribution will be, since g = Σ1/(1+λ), perhaps this combination of variables somehow eliminates length dependence as a parameter at the critical point? Perhaps it is the first eigenvalue, ν1, or λ1 or something that is invariant at the critical point, while in the metallic regime it gets pushed into the origin, and in the insulating regime it gets pushed away from the origin?

**Eigenvector correlations**

Both the ν’s and eigenvector correlations, discussed below, ought to have peculiar properties in each of the three phases.  Eigenvector amplitudes |υmn|2 is ~ fraction of current in channel n that propagates into channel m.  In the weak disorder case, we expect|υmn|2 ~ 1/N.  For strong disorder, we expect |υmn|2 ~ δ{m,n},{α,β}, where α, β are random indices .  Therefore the important-later-matrix Kmn = Σa<|υma|2|υna|2>  would be ~ 1/N for weak disorder, but Kmn ~ δmn, for strong disorder.  Thus the diagonal elements of the K-matrix evince a behavior similar to ν­min itself, in the thermodynamic limit.  This suggests matrix γm≠n = 2Km≠n/Kmm might also be construed as an order parameter as it would be 1 in the Q1D/3D weak disorder regime, and approximately 0 in the insulating regime (thermodynamic limit).  Presumably at the critical disorder there would a fixed point.

**How does P(g) scaling fit in with SPS theory?**

Wbat does this mean precisely? That G(bL) = f(b,G(L)). This would imply G(L) = f(L,G(1)). And so G(L) depends on L and G(1) alone. If G(1) > G\*, then it will scale towards conductor, and if G(1) < G\*, then towards insulator. This is equivalent to RG: dlnG/dlnL = β(G). So for rectangle, we need P(G;w,z,ℓ). Could solve for <G>(w,z,ℓ), and thereby write ℓ(<G>,w,z). Then for cube we’d have P(G;w,z,<G>) and so you could simply specify <G> at the given w,z and then get the probability distribution. Or could solve for <G> at a given configuration z=w=ℓ, say, and have ℓ(<G>), and then write P(G;w,z,<G>), though now <G> would be the conductance for that microscopic cube. How would we know if our P(G,w,z,ℓ) is consistent with the scaling law g(bL) = f(b,g(L))? We can get an equation: <G>(L,ℓ). But whether it’s consistent or not we wouldn’t know unless we formulated ∂ln<G>/∂lnL. Again, not sure if it’s *necessary* to be consistent with the scaling law.

It’s possible that P could depend on more parameters than just ℓ, or <G>, though, as long as <G> obtained from P(G) depended on only one? The other parameter(s) could affect ‘trivial’ parts of the distribution, say perhaps much higher moments that don’t affect the bulk of the distribution. I think consensus, though, is that P(G) does depend on just one parameter. Well but also remember things like Tr(λ) would seem to always grow with L (at least because # of channels is growing) and so it’d seem unlikely that the distribution would be truly stationary since P(g) must depend on all λ’s. But then, since these larger λ’s are insignificant to g, it does seem that P(g) will be effectively stationary. So to test for phase transition-capturing, we’ll probably need to calculate P(g), or <g>, itself.

Another question. What difference does it make if say g(L) = f(L,g1), rather than g(bL) = f(b,gL)? It seems to me that the former could just as easily accomodate the transition. The latter implies the former, though not vice versa. The latter *readily* explains why phase transition happens in 3D. There seems to be support for latter as well. But how would this evince itself in P(g), or P(λ)? Certainly they can only depend on one parameter, but that’s just the weaker former statement.

**Trying to reconcile SPS with GDMPK**

So a formula like g(L) = aLb would work, as: g(λL) = λbaLb = λbg(L). What about g(L) = L0 + L? In that case g(λL) = L0 + λL = L0+L + λL – L = g(L) + L(λ-1). And this is not a function of g(L), λ, alone. So this g(L) = L0 + L is not a ‘single parameter’ scaling function. Consider g(χ,L) = g0(L/χ)e-(L/χ) for instance. Then we cannot say that g(χ,bL) = f(g(χ,L),bL) I don’t think. Or say,



but,



for instance because if we set b = 2, then we’d have the implication:



But metallic conductance behaves this way: g = σLd-2 → g(bL) = σLd-2bd-2 = g(L)bd-2. And in insulating state: g = g0e-L/ξ → g(bL) = g0e-bL/ξ = (g0e-L/ξ)b/g0b = g(L)b/g0b. So hmmm…this would necessitate that g0 carries no information about impurities, but is perhaps the conductance quantum. I think it actually is.

How can a cube depend only on L/ξ? ξ can’t be localization length for metal, but is more or less for insulator. But single parameter scaling for cubes requires P(L/ξ) where ξ(ℓ) is some characteristic length scale, which could be taken to be localization length if insulator and what, if metal?



where ξ = λF2/ℓ here. And for cube,



And in insulating regime, we have:



and for cube,



How would the transition occur? Seems that



But how per se does this follow? Or does it? Consider ODE:



So now g depends on two parameters.



Does it follow from here the single parameter thing? g = e2e-L/ξ is consistent with it, but doesn’t require. Maybe its only an asymptotic formula? In our case we had P(g) = P(g, L/ξ, Lz/ξ). Are these equivalent? Well,



so yes. How would the transition occur? Still, it seems to me that we should have:



where χ = something like λF/ℓ. Perhaps in asymptotic regime, I can imagine. Consider:



it seems that setting L = L0, then conductance should only depend on the initial L0 and its conductance at that point. Can make L0 = *ξ*, and so,



But this doesn’t mean that g(L) depends only on L/ξ.



Now the GDMPK starts at zero length, which presumes all λ = 0, and so ballistic conductance. So this would be g at z = 0, and g = N, although he Suslov suggests that we really need to be using semi-transparent boundaries. Does this matter? So we should have, for Q1D,



Can this describe either situation? P(λ;χ,L,z). And when χ < 1, L = z evolves to insulator, but χ > 1, L = z evolves to metal? This must be true for any L. So in 1D, we have P(λ;χ,z). χ is just ℓ, and λF doesn’t even

**How would GDMPK describe a phase transition?**

So it seems that numerous people have worked out the moments of the conductance in the Q1D regime Lz << Nℓ = AkF2ℓ = L2ℓ/λF2. And N >> 1. Not sure how this relates to a cubic metal, but if Lz = L, then we’d have: L << L2ℓ/λF2 → 1 << Lℓ/λF2 → L >> λF2/ℓ, which could be done of course. Anyway, results coincide with expected metallic results.

So in general we find that if we have a multiplicative recursion relation Xm+1 = XmWm, then Xm will follow a ln-normal distribution – meaning the P(lnX) = Gaussian roughly. Moreover we will find that <lnX> and <lnX>2 both grow linearly with n. Note that this is roughly a classical picture of conduction too, though, right? In any event, this corresponds to the Q1D result.

Is it possible to get some sort of phase-transitiony stuff from equation. Consider our simple model.



Then we may write Kmm = 4ℓ/ξ ≡ K, and γm≠n = ξ/8LT ≡ γ. And,



So the independent parameters are LzK ~ Lz/ξ, and γ ~ ξ/LT. So we’d have P(xn, Lz/ξ, ξ/LT). We could take ξ as the independent parameter. So one could imagine that the exponents’ development would depend on ξ. If ξ > LT, then decreasing Lz would make them go to Q1D state. Otherwise, they’d go to 3D state. Basically ξ/LT would be analogous to our *w* parameter somewhere. LT = Lz, then we just have P(xn,L/ξ). But here there are no scaling possibilities, only a single definite outcome. So basically, we need a third independent parameter: P(xn, Lz, LT, ℓ), say. It is easy to see we have this at least in the Cartesian representation… σab = φ/Aℓkakb. Now,



So it would seem that the only parameters we’ll have are basically Lz/ℓ, and N, or Lz/ℓ and AkF2 = L2kF2. So we have two parameters z/ℓ, and L2kF2. Or in other words:



And cube would be:



Can that work? Can that bifurcate?



Sort of? Propose:



So it *can* clearly bifurcate. Could also write this as:



but in this case ξ would be allowed to be negative. So maybe not in the spirit of things. But at least in either asymptotic regime, it can be written as f(L/ξ). And we’ll note that this is a function of a single parameter. Could I write a function that splits between power and exponential?



So we sort of get what we want here. How does this reconcile with the single parameter scaling theory deal? The scaling equation seems to allow two parameters: L0, g0. And say one is standard L0 = λF ~ lattice spacing. Then still we need another: g0, which we could write in terms of dimensionless conductance g0 = e2ℓ. And so I think it’s acceptable to say that Pcube(g) = f(λF,ℓ,L), and that λF and ℓ don’t necessarily need combine into a single parameter. And so then Prectangle(g) = f(λF,ℓ,L,z). And it seems suportable, but not necessary, that in the asymptotic regime it turns into single parameter dependence. Not sure about that. In any case, I could’ve modified my example to make it a function of a single parameter. Can we calculate a β(g) for this?



Seems questionable. But consider the simpler scaling relationship (which scales to constant, rather than L, in the less than critical disorder case)



and this can be done. All we have to do is solve for that single parameter (1-χ)L/λF in terms of g.



And so we’d have:



It’s kind of got a kink in it at g = 1, but anyway…

**What can DMPK do that NLσM can’t?**

NLσM can get moments in 2+ε dimensions. But Shapiro makes a remark about how we cannot always reconstruct the probability distribution from the moments. Well indeed, because sometimes the moments diverge, like for the Cauchy distribution. Only P(g) can address issues like the singularity, if there is one (seems doubtful in any case). We can easily write an equation for different statistical models too. What are applications in other areas: phonons, photons?